The basic definition of an integral, as follows:

\[ \int_a^b f(x)dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f(w_i) \]

Let \( n \) be the number of bars required to fill the area within your interval. Then \( \Delta x_i = \frac{b-a}{n} \) is the width of the bars (constant over the interval). So, as we decrease the width of the bars \( \Delta x_i \) over the interval, we correspondingly increase the number of bars \( n \) required to fill the space. Hence these values are inversely proportionate. We define these bars so that there is some point on the x-axis \( w_i = a + \frac{i(b-a)}{n} \) for which the surface of the graph crosses through each bar. We denote this corresponding coordinate on the y-axis as \( f(w_i) \). Note: we can convert our sigma notation when considering this integral approximation.

For instance, below we consider the function \( f(x) = x^4 - 5x^2 + 3 \) over the interval \([5, 9]\). Solving for the integral we know that:

\[ \int_5^9 (x^4 - 5x^2 + 3)dx = \left( \frac{1}{5}x^5 - \frac{5}{3}x^3 + 3x + C \right) \bigg|_5^9 \]

\[ = \frac{1}{5}(9)^5 - \frac{5}{3}(9)^3 + 3(9) - \left( \frac{1}{5}(5)^5 - \frac{5}{3}(5)^3 + 3(5) \right) \approx 10190 \]

In the first graph \( n = 4, \Delta x_i = \frac{b-a}{n} = 1 \), hence for each of the 4 \( w_i \): (5.5, 6.5, 7.5, 8.5). There exists a corresponding \( f(w_i) \): (766.8, 1576.8, 2885.8, 4861.8), which we sum to approximate our integral \( \sum_{i=1}^{4} f(w_i)(1) \approx 10091 \). Close, but not close enough.
Now for the second graph $n = 8$, $\Delta x_i = 0.5$, hence for each of the 8 $w_i$: $(5.25, 5.75, 6.25, 6.75, 7.25, 7.75, 8.25, 8.75)$. There exists a corresponding $f(w_i)$: $(624.9, 930.8, 1333.6, 1851.1, 2503.0, 3310.2, 4295.2, 5482.0)$, which we enter into our equation $\sum_{i=1}^{8} f(w_i)(0.5) \approx 10165$

Finally, we let $n = 50$, $\Delta x_i = 0.1$, hence for each of the 50 $w_i$ (too many to list). There exists a corresponding $f(w_i)$, which we enter into our equation $\sum_{i=1}^{50} f(w_i)(0.1) \approx 10189$. A much better approximation.
Integration and Substitution

The basic rules/tips for integration: \[ \int_{a}^{b} f(x) \, dx \]

- Your function must be continuous and differentiable over your interval \([a, b]\). When you have a point of discontinuity, your function is no longer bound to a specific value for the associated \(x\). Hence, the integral of your function is not bound at this point.
  
  Note: an exception occurs when you require 2 functions to define your line and they do not form a smooth function (i.e. gaps in between). Unless both functions are defined at their connective point, your function is not considered bounded at that point.

Ex: The integral of \(f(x) = 3\) over \([1, 5] = \int_{1}^{5} 3 \, dx = (3x + C) \mid_{1}^{5} = 15 - 3 = 12\)

Consider the area of this box created in the \(x - y\) coordinate system. You have \(y = 3\) from \(x = 1\) to \(5\). So, the corresponding area of the box would be \(3 \times (5 - 1 = 4) = 12\).

- If \(f(x) = g(x) \pm h(x)\), then \(\int_{a}^{b} f(x) \, dx = \int_{a}^{b} g(x) \, dx \pm \int_{a}^{b} h(x) \, dx\)

- Likewise, if some \(x = c\) falls inside your interval \([a, b]\), then
  \[ \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \]

**NOTE:**

\[ \int_{a}^{b} f(x) g(x) \, dx \neq \int_{a}^{b} f(x) \, dx \times \int_{a}^{b} g(x) \, dx \]

Consider the same derivatives:

\[ f^1(x) g^1(x) = \left( \frac{d(f(x))}{dx} \right) \left( \frac{d(g(x))}{dx} \right) \neq \left( \frac{d(f(x)g(x))}{dx} \right) = (f(x)g(x))^1 \]

\[ \int_{a}^{b} \frac{f(x)}{g(x)} \, dx \neq \frac{\int_{a}^{b} f(x) \, dx}{\int_{a}^{b} g(x) \, dx} \]

Likewise, \[ f^1(x) = \frac{d(f(x))}{dx} \neq \left( \frac{d(f(x))}{g(x)} \right) = \left( \frac{f(x)}{g(x)} \right) \]

- Always consider **u-substitution**, when you are dealing with polynomials and exponents or exponents of trigonometric functions. Always consider a substitution that will simplify your integration. Remember that integration is backwards differential form, so use \(du\) to complete the chain rule. As a check the derivative of your integral should always give you same terms as the integral. \(\int du = u + C\)

Ex: So, if you have \((2x + 1)^{1/2}\) the derivative is \((2x + 1)^{-1/2} \, dx\). If we let \(u = 2x + 1\) then the corresponding \(du = 2 \, dx\). And the derivative of \(u^{1/2}\) is \(d(u^{1/2}) = \frac{1}{2} u^{-1/2} \, du = \frac{1}{2} (2x + 1)^{-1/2} (2 \, dx) = (2x + 1)^{-1/2} \, dx = \frac{dx}{(2x + 1)^{-1/2}}\)

Ex: Another substitution: \(d(x^2 + 1)^2\) becomes \(4x(x^2 + 1) \, dx\). We need \(u = x^2 + 1\), then corresponding derivative of \(u\) is \(du = 2x \, dx\). And the derivative of \(u^2\) is \(d(u^2) = 2udu = 2(x^2 + 1)(2x \, dx) = 4x(x^2 + 1) \, dx\)
When u-substitution doesn’t have all terms cancel, you may consider alternative methods. Such as, can you simplify your equation to eliminate a term in the denominator that may simplify your integration? For instance, when you expand your polynomial, can you integrate each term more easily?  

Ex: \( \int \frac{(10x+5)^2}{10x} \, dx \) We let \( u = 10x + 5 \) then \( du = 10 \ln 10 \, dx \), but there is no \( 10x \) term in the numerator to sub in \( du \), so we can’t make this substitution.

Let’s try simplifying the equation.

\[
\frac{10^2x+10^x+25}{10^x} = \frac{10^2x}{10^x} + \frac{10^x+1}{10^x} + \frac{25}{10^x} = 10 + 10 + 25(10^{-x})
\]

The only substitution that may be necessary is \( u = -x \). Now we add our 3 integrals. \( \int 10^x \, dx + \int 10 \, dx - 25 \int 10^u \, du \)

Let’s recall our table of derivatives (Note: the derivative of your integral should be equal to the function inside the integral).

Also \( uv + C = \int v \, du + \int u \, dv \)

Ex: Solve \( \int x^3 \, dx \)

Let \( u = x \), so we calculate \( du = dx \) and \( v = 3^2x \), with \( dv = 2 \ln 3(3^2x) \, dx \)

We recognize that we need to solve \( \int u \, dv \) with some cancellations for the constant on \( dv \). Let’s rewrite our integration by parts equation with the current substitutions in place and rearrange to solve.

\[
x^{3^2} + C_1 = \int 3^{2^2} \, dx + 2 \ln 3 \int x^{3^2} \, dx \quad \text{with} \quad \int 3^{2^2} \, dx = \frac{1}{2 \ln 3} 3^{2^2} + C_2
\]

\[
\Rightarrow \frac{1}{2 \ln 3} [x^{3^2} + C_1 - \frac{1}{2 \ln 3} 3^{2^2} - C_2] = \int x^{3^2} \, dx
\]

The integral shown below is definitely going to require a trigonometric substitution, because if we let \( u = 1 - x^2 \) then we need to have \( du = 2x \, dx \), since \( 2x \) is not available in this equation; we resort to alternative methods.
Ex: \( f(x) = 1 - x^2 \) over the interval \([-1, 1]\).

\[
\int_{-1}^{1} (1 - x^2) \, dx = (x - \frac{1}{3} x^3 + C) \bigg|_{-1}^{1} = [(1(1) - \frac{1}{3}(1)^3] - [1(-1) - \frac{1}{3}(-1)^3] = \frac{4}{3}
\]

Now, let \( f(x) = \sqrt{1 - x^2} \) over the interval \([-1, 1]\).

\[
\int_{-1}^{1} \sqrt{1 - x^2} \, dx \quad \text{Our trigonometric identities, in this case } \sin^2 x + \cos^2 x = 1, \text{ will allow us to simplify this expression. We use this identity and let } x = \sin u.
\]

**Note:** when you make a substitution for \( x \), you are also changing the interval over which this integral is to be evaluated. This becomes most evident with trigonometric substitution. Here we notice visually that the graph of \( f(x) \) over \([-1, 1]\) is equivalent to \( \sin u \) over \( u = \sin^{-1} x \rightarrow [\frac{-\pi}{2}, \frac{\pi}{2}] \).

This time we solve for the derivative of \( x \) (reverse to the regular substitution method). So, \( dx = \cos u \, du \). Now we substitute our \( x \) and \( dx \) back into the original equation.

\[
\int_{-1}^{1} \sqrt{1 - x^2} \, dx = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \sin^2 u} \cos u \, du = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^2 u} \cos u \, du = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^2 u \, du
\]

Again, we are left with an integral that we cannot evaluate directly (i.e. without some type of substitution). At least we have numerous trigonometric identities at our disposal. Such as \( \cos 2u = \cos^2 u - \sin^2 u \) & \( \sin^2 u + \cos^2 u = 1 \) provide us with the substitution \( \cos^2 u = \frac{1}{2} \cos 2u + \frac{1}{2} \), which is integrable, so we enter the substitution back into the equation.

\[
\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^2 u \, du = \frac{1}{2} \left[ \cos 2u + 1 \right] \, du
\]

\[
= \frac{1}{2} \left( \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos 2u \, du + \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} du \right) = \frac{1}{2} \left[ \frac{1}{2} \sin 2u + u \right] \bigg|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2} \left[ \frac{1}{2} \sin(2 \times \frac{\pi}{2}) + \frac{\pi}{2} - \frac{1}{2} \sin(2 \times -\frac{\pi}{2}) - \frac{\pi}{2} \right] = \frac{1}{2} \left[ (0 + \frac{\pi}{2}) - (0 - \frac{\pi}{2}) \right] = \frac{\pi}{2}
\]

You may recognize this equation, as the positive half of the equation of a circle \( x^2 + y^2 = 1 \rightarrow y = \pm \sqrt{1 - x^2} \). The area of a circle is given by \( A = \pi r^2 \), in our case \( r = 1 \) and we only want the area of half of the circle, hence \( A = \frac{\pi}{2} \) and we have double checked our answer.
Some identities may not appear useful in their current form; however we can also combine them together to find a solution. For instance, the double angle formulas may not be the most obvious choice, unless we consider them half angle formulas.

\[
\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}; \tan 2x = \frac{2 \tan x}{1 - \tan^2 x}
\]

\[
\Rightarrow \tan x = \frac{1}{2} \tan 2x \left( 1 - \frac{1 - \cos 2x}{1 + \cos 2x} \right) = \frac{1}{2} \tan 2x \left( \frac{2 \cos 2x}{1 + \cos 2x} \right)
\]

\[
2 \cos^2 x = 1 + \cos 2x \Rightarrow 2 \cos^2 x \tan x = \cos 2x \tan 2x
\]

\[
\csc x = \frac{1}{\sin x} = \frac{1}{\cos x \tan x} = \frac{1}{2 \cos^2 \frac{x}{2} \tan \frac{x}{2}} = \frac{\sec^2 \frac{x}{2}}{2 \tan \frac{x}{2}}
\]

\[
\int \csc x \, dx = \int \frac{\sec^2 \frac{x}{2}}{\tan \frac{x}{2}} \, dx \quad \text{If } u = 2 \tan \frac{x}{2}, \text{ then correspondingly } du = \sec^2 \frac{x}{2} \, dx
\]

\[
= \int \frac{1}{u} \, du = \ln u + C = \ln \left| \tan \frac{x}{2} \right| + C = \ln |\csc x - \cot x| + C
\]

\[
\csc x - \cot x = \frac{1 - \cos x}{\sin x} \Rightarrow \text{Double angle formulas } = \frac{2 \sin^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \tan \frac{x}{2}
\]

Integrating fractions may not be solved with simple substitution rules. In such cases, as long as exponent in the numerator is smaller than the denominator, you can use partial fractions to simplify your integral. This process is a reversal of finding a common denominator. The strategy for these equations is finding the lowest possible denominator or stop when you have a differentiable function. We need to use several methods (even those learned in early high school) to factor these denominators.

Ex: Solve \[
\int \frac{\frac{1}{2}x^2 + 2x + 1}{x^4 + 4} \, dx
\]

**Step 1** - Factor your denominator \(x^4 + 4\) Recall quadratic equations before the time of the quadratic formula.

\[
x^4 + 4x^2 - 4x^2 + 4 = (x^2 + 4x^2 + 4) - 4x^2 = (x^2 + 2)^2 - (2x)^2
\]

\[
= (x^2 + 2 + 2x)(x^2 + 2 - 2x)
\]

Now with our current factored form, we attempt to find values for \(A\) and \(B\) that will satisfy the split factored fractions, such that:

\[
(Ax + B)(x^2 + 2 - 2x) + (Cx + D)(x^2 + 2 + 2x) = 1
\]

\[
Ax^3 + 2Ax^2 + Bx^2 + 2B - 2Bx + Cx^3 + 2Cx + 2Cx^2 + Dx^2 + 2D + 2Dx = \frac{1}{2}x^2 + 2x + 1
\]

**Step 2** Sum the coefficients of each power of \(x\):

\[
\Rightarrow \quad A + C = 0
\]

\[
\Rightarrow \quad -2A + B + 2C + D = \frac{1}{2}
\]

\[
\Rightarrow \quad 2A - 2B + 2C + 2D = 2
\]

\[
\Rightarrow \quad 2B + 2D = 1
\]
1 0 1 0 0 
-2 1 2 1 1/2 
2 -2 2 2 2 
0 2 0 2 1 

\[ R_1 + \frac{1}{2}R_2 \]

0 1/2 2 1/2 1/4 
-2 1 2 1 1/2 
2 -2 2 2 2 
0 2 0 2 1 

\[ R_1 - \frac{1}{4}R_4 \]

0 0 1 0 0 
-2 1 2 1 1/2 
2 -2 2 2 2 
0 2 0 2 1 

\[ R_3 + R_2 \]

0 0 1 0 0 
-2 1 2 1 1/2 
2 -2 2 2 2 
0 2 0 2 1 

\[ R_3 + \frac{1}{2}R_4 \]

0 0 1 0 0 
-2 1 2 1 1/2 
0 0 0 1 3/4 
0 2 0 2 1 

\[ R_3 + \frac{1}{2}(R_4 - 2R_3) \]

0 0 1 0 0 
-2 1 2 1 1/2 
0 0 0 1 3/4 
0 1 0 0 -1/4 

\[ R_1 - \frac{1}{2}(R_2) \]

1 0 0 0 0 
-1/2 1 2 1 1/2 
0 0 0 1 3/4 
0 1 0 0 0 

\[ R_3 - \frac{1}{2}R_4 \]

0 1 0 0 0 
-1/4 1 2 1 1/2 
0 0 0 1 3/4 
0 1 0 0 0 

\[ R_4 - \frac{1}{2}R_3 \]


\[ A = 0, B = -\frac{1}{4}, C = 0, D = \frac{3}{4} \]

Assuming that we will rewrite our new function as:

\[
\int \frac{\frac{1}{2}x^2 + 2x + 1}{x^4 + 4} \, dx = \int \frac{B \, dx}{(x^2 + 2 + 2x)} + \int \frac{D \, dx}{(x^2 + 2 - 2x)}
\]

\[
= \int \frac{-1}{4 (x^2 + 2 + 2x)} \, dx + \int \frac{3}{4 (x^2 + 2 - 2x)} \, dx
\]

Simplified, but not solvable in this form, so we must factor each denominator further

\[(x^2 + 2 + 2x) = ((x + 1)^2 + 1^2)\]
\[(x^2 + 2 - 2x) = ((x - 1)^2 + 1^2)\]

We should recognize the integral/derivative related to this function.

\[ d \left( \tan^{-1} x \right) = \frac{1}{x^2 + 1} \]

And substitution back into our integral.

\[ \Rightarrow = -\frac{1}{4} \int d(\tan^{-1}(x + 1)) + \frac{3}{4} \int d(\tan^{-1}(x - 1)) \]

\[ = -\frac{1}{4} \tan^{-1}(x + 1) + \frac{3}{4} \tan^{-1}(x - 1) + C \]